

# ORIGINS OF DIFFUSION

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**ABSTRACT.** We consider a dynamical system consisting of subsystems indexed by a lattice. Each subsystem has one conserved degree of freedom ("energy") the rest being uniformly hyperbolic. The subsystems are weakly coupled together so that the sum of the subsystem energies remains conserved. We prove that the long time dynamics of the subsystem energies is diffusive.

## 1. DIFFUSION FROM CONSERVATIVE DYNAMICS

One of the fundamental problems in deterministic dynamics is to understand the microscopic origin of dissipation and diffusion. On a microscopic level a physical system such as a fluid or a crystal can be described by a Schrödinger or a Hamiltonian dynamical system with a macroscopic number of degrees of freedom. Although the microscopic dynamics is reversible in time one expects dissipation to emerge in large spatial and temporal scales e.g. in the form of diffusion of heat or concentration of particles.

To fix ideas, consider a Hamiltonian dynamical system i.e. a Hamiltonian flow on a symplectic manifold  $M$ . For the present purpose it suffices to consider  $M = \mathbb{R}^{2n}$  with position and momentum coordinates  $q, p \in \mathbb{R}^n$ . The Hamiltonian flow  $\phi_t \in \text{Diff} M$  generated by the vector field  $(\partial_p H, -\partial_q H)$  where  $H : M \rightarrow \mathbb{R}$  is the Hamiltonian or energy function preserves the energy

$$H \circ \phi_t = H$$

i.e. the flow preserves the constant energy sets  $M_E = \{(q, p) : H(q, p) = E\}$ .

On the other hand, the simplest diffusion process is given by the heat equation

$$(1) \quad \partial_t E(t, x) = \kappa \Delta E(t, x)$$

and the associated semigroup  $\psi_t = e^{\kappa t \Delta}$ . Unlike for the reversible  $\phi_t$  where  $\phi_{-t} = \phi_t^{-1}$ ,  $\psi_t$  has no inverse and describes dissipation. Physically, the energy function  $E(t, x)$  describes a macroscopic energy density i.e. a coarse grained function of microscopic dynamical variables, the positions and momenta of the underlying Hamiltonian dynamics. The question we wish to pose is how does this dissipative dynamics  $\psi_t$  arise from the conservative one  $\phi_t$ .

A concrete physical system where diffusion occurs is a fluid. In classical mechanics this is microscopically modeled by a Hamiltonian system whose flow gives the trajectories of the fluid particles  $(q_i(t), p_i(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$ ,  $i = 1 \dots N$ . A typical Hamiltonian function is given by

$$(2) \quad H(q, p) = \sum_i \frac{p_i^2}{2m} + \sum_{ij} V(q_i - q_j)$$

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consisting of the kinetic energy of the particles of mass  $m$  and a pair potential energy of interaction of the particles. Let the energy of the  $i$ :th particle be defined as

$$(3) \quad e_i = \frac{p_i^2}{2m} + \frac{1}{2} \sum_{j \neq i} V(q_i - q_j)$$

so that  $H = \sum_i e_i$ . We can then define the energy density as the distribution

$$(4) \quad E(t, x) = \sum_i e_i \delta_{q_i(t)}(x)$$

where  $\delta_q$  is the Dirac mass at  $q$ . Since  $\int E(t, x) dx = \sum_i e_i = H$  and  $\dot{H} = 0$  one concludes

$$(5) \quad \dot{E}(t, x) = \nabla \cdot J(t, x)$$

for a certain distribution, the energy current, depending on  $q(t), p(t)$ . Eq. (5) is a local conservation law deduced from the global energy conservation. In the case of the fluid, there are two other similar local conservation laws related to global momentum and particle number conservation laws. This leads to a richer set of macroscopic laws in the case of the fluid than the diffusion equation for the energy (in particular these include the Navier-Stokes equations).

## 2. COUPLED OSCILLATORS

Thus, to understand the origins of diffusion one should look for systems with just one local conservation law eq. (5). There has been a lot of work in recent years around these questions in the context of **coupled dynamics** i.e. dynamical systems consisting of elementary systems indexed by a  $d$ -dimensional lattice  $\mathbb{Z}^d$ . The total energy  $E$  of the system is a sum  $\sum_x E_x$  of energies  $E_x$  which involve the dynamical variables of the system at lattice site  $x$  and nearby sites. The physical situation to keep in mind is then thermal conduction in a crystal lattice (i.e. a solid).

Two types of systems have been considered. In the first case at each lattice site we have an oscillator and the oscillators at neighboring sites are coupled together. Typically one considers the system where the forces are weakly anharmonic. In the second case at each lattice site one puts a chaotic system and weakly couples the neighboring systems. Let us start with the former case.

The setup resembles that of the fluid above, but now the "particle" positions  $q_x$  are indexed by the lattice,  $x \in \Lambda \subset \mathbb{Z}^d$  where  $\Lambda$  is a finite subset, say a cube, and they describe the deviation of an atom from its equilibrium position at  $x$ . A simple classical mechanical model for this is a system of coupled oscillators

$$(6) \quad H_\Lambda(q, p) = \sum_{x \in \Lambda} \left( \frac{p_x^2}{2m} + U(q_x) \right) + \sum_{|x-y|=1} V(q_x - q_y)$$

where  $U$  is a *pinning* potential which we assume tending to infinity as  $|q| \rightarrow \infty$ . The potential  $V$  describes interaction of the atoms in nearest neighbor lattice sites and is taken attractive. A challenging model is obtained already by taking

$$(7) \quad V(q) = q^2, \quad U(q) = q^2 + \lambda|q|^4$$

and further simplifying by taking  $q_x \in \mathbb{R}$  instead of  $\mathbb{R}^d$ . Then a lattice version of eq. (5) holds with the current given by

$$(8) \quad J_\mu(x) = -\frac{1}{2}(p_{x+\mu} + p_x)V'(q_{x+\mu} - q_x).$$

In what sense should we expect the conservative dynamics (5) give rise to a diffusive one as in eq. (1)? The answer is that this should happen for *typical* initial conditions  $(q(0), p(0)) \in M_\Lambda$  with respect to a specific measure on the phase space  $M_\Lambda := \mathbb{R}^{2|\Lambda|}$  and under a proper *scaling limit*.

Recall first that the Hamiltonian dynamics preserves the Lebesgue measure  $m_\Lambda$  on  $M_\Lambda$ . Since also  $H_\Lambda$  is preserved so is the Gibbs measure (or *equilibrium measure*)

$$\mu_{\beta\Lambda} = Z_\Lambda^{-1} e^{-\beta H_\Lambda} m_\Lambda$$

where  $\beta > 0$  as well as its (thermodynamic) limit  $\mu_\beta = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \mu_{\beta\Lambda}$ . Let us now replace the (inverse) temperature parameter  $\beta$  by a spatially varying one. Let  $b \in C_0^\infty(\mathbb{R}^d)$  and  $\beta > \|b\|_\infty$ . Write as in the fluid case

$$H_\Lambda = \sum_{x \in \Lambda} e_x$$

,  $e_x$  describing the energy contributed by the oscillator at  $x$ . Pick a scaling parameter  $L \in \mathbb{N}$  and set  $\beta_L(x) = \beta + b(x/L)$ . Let  $\mu^{(L)}$  be the thermodynamic limit of the measure

$$Z_{L,\Lambda}^{-1} e^{-\sum_{x \in \Lambda} \beta_L(x) e_x} m_\Lambda.$$

Construction of this limit poses no problems if  $\lambda \geq 0$  in eq. (7) is small enough.  $\mu^{(L)}$  is *not* invariant under the dynamics which maps it to  $\mu_t^{(L)} = \mu^{(L)} \circ \phi_t^{-1}$ . However, one expects that as  $t \rightarrow \infty$  there is *return to equilibrium* i.e.  $\mu_t^{(L)} \rightarrow \mu_\beta$ . The diffusion equation is expected to govern this process in the following sense.

Let  $f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$  and consider the random variables

$$(9) \quad e_L(f) = L^{-d-2} \sum_{(t,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d} f(t/L^{-2}, x/L) e_x(q(t), p(t)).$$

The statement of the *hydrodynamic limit* is then: with probability one in the sequence of measures  $\mu^{(L)}$ ,  $e_L(f)$  converges to  $\int f(t, x) E(t, x) dt dx$  where  $E$  is the solution to the nonlinear diffusion equation

$$(10) \quad \partial_t E = \nabla \cdot (\kappa(E) \nabla E)$$

where  $\kappa(E)$  is a smooth positive function. The initial condition  $E(0, \cdot)$  is determined by the function  $b$ . Thus upon coarse graining and scaling the equation (5) turns to eq. (10), *almost surely* in the initial conditions of the underlying microscopic variables.

The proof of the hydrodynamic limit in our model is beyond present mathematical techniques. The existing techniques require the presence of plenty of noise in the system. A simpler problem would be to establish the *kinetic limit*. This is a *weak anharmonicity* limit. We replace  $\lambda$  in eq. (7) by  $\lambda/\sqrt{L}$  and consider the measures  $\mu_{L,t}^{(L)}$ . As  $L \rightarrow \infty$  we expect these measures to become gaussian whose covariance upon spatial scaling satisfies a Boltzman equation. More precisely, denote  $(q_x, p_x)$  by  $\phi(x)$ . Then it is conjectured that

$$(11) \quad \lim_{L \rightarrow \infty} \int \phi(Lx + y) \phi(Lx - y) \mu_{L,t}^{(L)}(d\phi) = G(t, x, y)$$

exists and the Fourier transform of  $G(t, x, y)$  in  $y$ ,  $\hat{G}(t, x, k)$  satisfies the so called phonon Boltzman equation

$$(12) \quad \partial_t \hat{G}(t, x, k) + \nabla \omega(k) \cdot \nabla \hat{G}(t, x, k) = I(\hat{G}(t, x, \cdot))$$

where  $I$  is a nonlinear integral operator and  $\omega(k)^2$  is the Fourier transform of the lattice operator  $2(-\Delta + 1)$ , see [1]. Proof of these statements is still open and a considerable challenge (for some progress see [2]). Derivation of a hydrodynamic equation of the type (10) from the Boltzman equation (12) has been carried out [3], see also [4] where an attempt to go beyond the kinetic limit was carried out.

### 3. COUPLED CHAOTIC FLOWS

A second class of models deals with a complementary situation of *weakly coupled chaotic* systems [5], [6], [7]. The setup is as follows. Let  $(M, H)$  be a Hamiltonian system i.e.  $M$  is a symplectic manifold and  $H : M \rightarrow \mathbb{R}$ . Let, for each  $x \in \mathbb{Z}^d$   $(M_x, H_x)$  be a copy of  $(M, H)$ . Let  $h : M \times M \rightarrow \mathbb{R}$  and for each  $x, y \in \mathbb{Z}^d$ ,  $|x - y| = 1$  let  $h_{xy} : M_x \times M_y \rightarrow \mathbb{R}$  be a copy of  $h$ . Let  $\Lambda \subset \mathbb{Z}^d$  be finite and  $M_\Lambda = \times_{x \in \Lambda} M_x$ . The coupled flow is the one on  $M_\Lambda$  generated by the Hamiltonian

$$(13) \quad H_\Lambda = \sum_{x \in \Lambda} H_x + \sum_{|x-y|=1} \lambda h_{xy}.$$

Of course, the coupled oscillators of the previous section are of this form. There, the system  $(M, H)$  is integrable, and the diffusive dynamics is the consequence of coupling and anharmonicity. In the present discussion we wish to take  $(M, H)$  chaotic. Examples are Anosov systems or billiard systems. E.g. in the former case the flow on  $M$  generated by  $H$  has  $\dim M - 2$  non-zero Lyapunov exponents and two vanishing ones corresponding to the Hamiltonian vector field and  $\nabla H$ .

When the coupling parameter  $\lambda$  is zero  $(M_\Lambda, H_\Lambda)$  has  $2|\Lambda|$  vanishing Lyapunov exponents. For  $\lambda \neq 0$  one expects that for a large class of perturbations  $h$  the only constant of motion is  $H_\Lambda$  and the system has only two vanishing exponents. However, zero should be near degenerate for the Lyapunov spectrum and these long time scale motions should be at the origin to diffusion in the  $\Lambda \rightarrow \mathbb{Z}^d$  limit.

Rigorous results on such Hamiltonian systems are rare: in [5] ergodicity is proved in a one dimensional model. However, it seems very difficult to get hold of the Lyapunov spectrum and it is far from obvious how such knowledge would turn into a proof of diffusion in these systems. I want to argue that a more fruitful approach is to study the local energy conservation law (5) and try to show that the chaotic degrees of freedom act there like a noise that redistributes locally the energy. To probe such an idea it is useful to turn to a discrete time version of our model i.e. to study iteration of a map rather than a flow.

### 4. COUPLED CHAOTIC MAPS

A discrete time version of the coupled flow setup of the previous section is called a Coupled Map Lattice (CML). Now the local dynamical system is a pair  $(M, f)$  where  $M$  is a manifold and  $f : M \rightarrow M$ . Again for each  $x \in \mathbb{Z}^d$   $(M_x, f_x)$  is a copy of  $(M, f)$  and  $(M_\Lambda, f_\Lambda)$  with  $f_\Lambda = \times_{x \in \Lambda} f_x$  is the product dynamics. The CML dynamics is a suitable local perturbation of the product dynamics.

Our choice of  $M$  and  $f$  is motivated by the coupled chaotic flows discussed before. A discrete time version (say given by a Poincare map) of a billiard or Anosov flow has one vanishing Lyapunov exponent corresponding to the conserved energy and the remaining ones nonzero. We model such a situation by taking for the local dynamics the manifold of form  $M = \mathbb{R}_+ \times N$  with  $N$  another manifold. Let us denote the variables at the lattice site  $x \in \mathbb{Z}^d$  by  $(E(x), \theta(x)) \in \mathbb{R}_+ \times N$ . We call the non-negative variables  $E$  energy and postulate them to be conserved under the local dynamics:

$$(14) \quad (E(x), \theta(x)) \rightarrow (E(x), g(\theta(x), E(x)))$$

for each  $x \in \mathbb{Z}^d$ .

$\theta \in N$  are the fast, chaotic variables. In the billiard case the dynamical system  $\theta \rightarrow g(\theta, E)$  is uniformly hyperbolic for any fixed  $E$ . We will model this situation by taking  $g(\theta, E) = g(\theta)$  a fixed chaotic map, independent of  $E$ . Examples are  $N = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$  and  $g$  an expansive circle map, e.g  $g(\theta) = 2\theta$  and  $N = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  and  $g$  a hyperbolic toral automorphism.

We should stress that the  $E$  independence is the most serious simplification in this setup. In a realistic Hamiltonian system, such as the billiards the  $E$  dependence of  $g$  can not be ignored. Indeed, it is obvious that as  $E \rightarrow 0$  the Lyapunov exponents of  $g(\cdot, E)$  also tend to zero since  $E$  sets the time scale.

The CML dynamics is a perturbation of the local dynamics (14). Let us use the same notation  $(E, \theta) \in M_\Lambda = \mathbb{R}_+^\Lambda \times N^\Lambda$ . Then  $F : M_\Lambda \rightarrow M_\Lambda$  is written as

$$(15) \quad F(x, E, \theta) = (E(x) + f(x, E, \theta), g(\theta(x)) + h(x, \theta)).$$

Here  $f$  and  $h$  are small local functions of  $(E, \theta)$  i.e. they depend weakly on  $(E(y), \theta(y))$  for  $|x - y|$  large as we will specify later.

$f$  is however constrained by the requirement that the *total energy*  $\sum_x E(x)$  is conserved. This follows if

$$\sum_x f(x, E, \theta) = 0$$

for all  $E, \theta$ . A natural way to guarantee this is to consider a "vector field"  $\mathbf{J}(x) = \{J^\mu(x)\}_{\mu=1, \dots, d}$  and take

$$(16) \quad f(x, E, \theta) = (\nabla \cdot \mathbf{J})(x, E, \theta) := \sum_\mu (J^\mu(x + e_\mu, E, \theta) - J^\mu(x, E, \theta))$$

With these definitions we arrive at the time evolution

$$(17) \quad E(t+1, x) = E(t, x) + \nabla \cdot \mathbf{J}(x, E(t), \theta(t))$$

$$(18) \quad \theta(t+1, x) = g(\theta(t, x)) + h(x, \theta(t)).$$

Note that (17) is a natural discrete space time version of (5). Let us discuss this iteration from a general perspective before making more specific assumptions of the perturbations.

## 5. FAST DYNAMICS

The iteration (18) of the chaotic variables is autonomous. We shall assume the perturbation  $h$  is  $C^1$  with the following locality property

$$(19) \quad |\partial_{\theta(y)} h(x, \theta)| \leq \epsilon e^{-a|x-y|}$$

and Hölder continuity property

$$(20) \quad |\partial_{\theta(y)} h(x, \theta) - \partial_{\theta(y)} h(x, \theta')| \leq \epsilon \sum_z e^{-a(|x-y|+|x-z|)} |\theta(z) - \theta'(z)|.$$

These properties guarantee [8] that the  $\theta$ -dynamics is *space-time mixing*. This means that the dynamics is defined in the  $\Lambda \rightarrow \mathbb{Z}^d$  limit and it has a unique Sinai-Ruelle-Bowen measure  $\mu$  on the cylinder sets of  $N^{\mathbb{Z}^d}$  which satisfies

$$(21) \quad \mathbb{E}(F(\theta(t, x))G(\theta(0, x))) - \mathbb{E}(F(\theta(t, y))\mathbb{E}G(\theta(0, y))) \leq Ce^{-c(t+|x-y|)}$$

for Hölder continuous functions  $F$  and  $G$ . Here  $\mathbb{E}$  denotes expectation in  $\mu$ .

We conclude that sampling  $\theta(0, \cdot)$  with  $\mu$  makes  $\theta(t, x)$  *random variables* which are exponentially weakly correlated at distinct space time points. Therefore  $\theta_x(t)$  acts as a *random environment* for the slow variable dynamics (17).

## 6. QUENCHED DIFFUSION

The previous discussion shows that we can view the current  $\mathbf{J}(x, E, \theta(t))$  in the slow variable dynamics (17) as a random field  $\mathbf{J}(t, x, E)$  which is exponentially weakly correlated in space and time. We may thus rephrase the problem of deriving diffusion in deterministic dynamics as that of *quenched diffusion in random dynamics*. We want to show that the random dynamical system

$$(22) \quad E(t+1, x) = E(t, x) + \nabla \cdot \mathbf{J}(t, x, E(t)) := \Phi(t, x, E(t))$$

has a diffusive hydrodynamical limit *almost surely* with respect to the SRB measure  $\mu$ . Let us inquire how this should come about and then list the assumptions we need for the actual proof.

Consider first the *annealed* problem, i.e. averaged equation (22):

$$E_x(t+1) - E_x(t) = \nabla \cdot \mathbb{E}[J(t, x, E(t))] := \nabla \cdot \mathcal{J}(x, E(t)).$$

where, by stationarity of  $\mu$ ,  $\mathcal{J}$  is time independent. Supposing that  $h$  and  $\mathbf{J}$  have natural symmetries under lattice translations and rotations we infer that  $\mathcal{J}$  vanishes at constant  $E$  and then locality assumptions of the type we assumed for  $h$  imply

$$\mathcal{J}(x, E) = \sum_y \kappa(x, y, E) \nabla E(y).$$

Hence the annealed dynamics is a *discrete nonlinear diffusion*

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t)$$

provided the diffusion matrix  $\kappa(E(t))$  is positive.

Let now

$$\beta(t, x, E(t)) = J(t, x, E(t)) - \mathcal{J}(x, E(t))$$

be the fluctuating part. Then slow dynamics becomes

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t) + \nabla \cdot \beta(t, E(t))$$

$$\mathbb{E} \beta(t, E) = 0$$

i.e. a *nonlinear diffusion* with a *random drift*. In a physical model one would expect  $\kappa(E(t))$  to be positive although not necessarily uniformly in  $E$ . If furthermore  $\beta$  turned out to be a small perturbation quenched diffusion might be provable. In what follows we will make such assumptions and then indicate how to establish diffusion.

Before stating the assumptions let us make one more reduction. It is reasonable to assume  $E = 0$  is preserved by the dynamics. This then implies  $\beta(t, 0) = 0$ . Let us study the linearization at  $E = 0$ :

$$(23) \quad E(t+1) - E(t) = \nabla \cdot \kappa(0) \nabla E(t) + \nabla \cdot (D\beta(0, t)E(t))$$

or, in other words

$$(24) \quad E_x(t+1) = \sum_y p_{xy}(t) E_y(t)$$

with

$$\sum_x p_{xy}(t) = 1.$$

Since  $E \geq 0$  we have  $p_{xy} \geq 0$  i.e.  $p_{xy}(t)$  are *transition probabilities of a random walk*.  $p_{xy}(t)$  is space and time dependent and random i.e. it defines a *random walk in random environment*.

## 7. RANDOM WALK IN NONLINEAR RANDOM ENVIRONMENT

Consider a random walk defined by the transition probability matrix  $p_{xy}(t)$  at time  $t$ .  $p(t) = p(t, \omega)$  is a taken random defined on some probability space  $\Omega$ . We suppose the law of  $p$  is invariant under translations in space and time. Define

$$(25) \quad \|E\| := \sup_x |E(x)| (1 + |x|)^{d+a}$$

for some  $a > 0$ . Let, at  $t = 0$ ,  $\|E\| < \infty$ . We say the walk defined by  $p$  is has a diffusive scaling limit if there exists  $C, \kappa$  such that *almost surely* in  $\omega$

$$(26) \quad \lim_{L \rightarrow \infty} \|L^d E(L^2 t, L \cdot) - C t^{-d/2} E_\kappa^*(\cdot / \sqrt{t})\| = 0$$

where  $E_\kappa^*(x) = e^{-x^2/4\kappa}$ . In other words

$$L^d E(L^2 t, Lx) \sim C t^{-d/2} e^{-x^2/4\kappa t}$$

as  $L \rightarrow \infty$ .

We prove this for a *non-linear perturbation* of RWRE. Let us state the assumptions for the random dynamical system eq. (22). We assume  $\Phi$  is  $C^2$  in  $\|E\|_1 < \delta$  and satisfies

*Positivity*:  $\Phi(E) \geq 0$  for  $E \geq 0$ .

*Conservation law*:

$$\sum_x \Phi(t, x, E) = \sum_x E_x$$

*Weak nonlinearity*:

$$\left| \frac{\partial^2 \Phi(t, x, E)}{\partial E_y \partial E_z} \right| \leq e^{-|x-y| - |x-z|}$$

Write the average map

$$\mathbb{E}\Phi(t, x, E) = \sum_y T(x - y) E_y + o(E).$$

*Ellipticity*:  $T$  generates a diffusive random walk on  $\mathbb{Z}^d$ .

Write

$$\Phi(t, x, E) - \mathbb{E}\Phi(t, x, E) := \nabla \cdot b(t, x, E).$$

*Weak correlations.* Assume

$$(27) \quad b(t, x, E) = \sum_{A \subset \mathbb{Z}^d \times [0, t]} b_A(t, x, E)$$

with

$$|b_A(t, x, E)| \leq \epsilon e^{-d((x, t) \cup A)}$$

and  $b_A, b_B$  are *independent* if  $A \cap B = \emptyset$ .

**Remark.** A representation of the form (27) arises from the model we have discussed above with the proviso that  $b_A, b_B$  are independent only in the case the  $\theta$  dynamics is local, i.e.  $h = 0$  in eq. (18). For the general  $h$  there is weak dependence that can be handled.

**Theorem 7.1.** Under the above assumptions and  $\delta, \epsilon$  small enough the random dynamical system  $\Phi_t$  is diffusive, almost surely in  $\omega$ .

## 8. RENORMALIZATION GROUP FOR RANDOM COUPLED MAPS

The proof of Theorem 7.1. [9] is based on a renormalization group method introduced in [10] and [11]. Let us introduce the scaling transformation  $S_L$ :

$$(28) \quad (S_L E)(x) = L^d E(Lx).$$

where  $L > 1$ . Fix  $L$  and define, for each  $n \in \mathbb{N}$ , *renormalized energies*

$$E_n(t) = S_{L^n} E(L^{2n} t).$$

We can then rephrase the scaling limit (26) as

$$\lim_{n \rightarrow \infty} L^{nd} E(L^{2n} t, L^n x) = \lim_{n \rightarrow \infty} E_n(t, x).$$

$E_n(t)$  inherits dynamics from  $E$ . We will call this the *renormalized dynamics*:

$$E_n(t+1) = \Phi_n(t, E_n(t)).$$

Explicitely we have

$$\Phi_n(t) = S_{L^n} (\Phi(L^{2n} t + L^{2n} - 1) \circ \dots \circ \Phi(L^{2n} t)) S_{L^{-n}}.$$

The dynamics changes with scale as

$$\Phi_{n+1} = \mathcal{R} \Phi_n$$

with

$$\mathcal{R} \Phi(t, \cdot) = S_L \Phi(t_{L^2}) \circ \dots \circ \Phi(t_1) S_L^{-1}$$

with  $t_1 = L^2 t$  and  $t_{L^2} = L^2(t+1) - 1$ .

$\mathcal{R}$  is the the *Renormalization group flow* in a space of random dynamical systems. We prove: almost surely the renormalized maps converge

$$\mathcal{R}^n f \rightarrow f^*$$

where the fixed point is *nonrandom and linear*:

$$f^*(E) = e^{\kappa \Delta} E.$$



Moreover, the renormalized energies converge almost surely to the fixed point

$$\|E_n(t, \cdot) - \frac{C}{t^{d/2}} E_\kappa^*(\cdot/\sqrt{t})\| \rightarrow 0$$

which is the diffusive scaling limit.

These results may be summarized by saying that both the randomness and the nonlinearity are *irrelevant* in the RG sense. Let us finish by sketching the reasons for this.

We start by considering the linear problem

$$D\Phi(t, x, 0)E = \sum_y p_{xy}(t) E_y.$$

Then  $DR\Phi = p'$  with

$$p'(t)_{xy} = L^d(p(L^2(t+1) - 1) \dots p(L^2 t))_{LxLy}.$$

Write

$$p_{xy}(t) = T(x - y) + \nabla_y \cdot c_{xy}(t)$$

with  $\mathbb{E}p = T$  and  $\mathbb{E}c = 0$ . Then, for  $p' = T' + \nabla c'$  we get

$$(29) \quad T'(x - y) = L^d T^{L^2}(Lx - Ly) + r(x - y)$$

where  $r$  is an expectation of a polynomial in  $c$ . For the noise we get

$$(30) \quad \nabla_x c'_{xy} = L^d \sum_{t=1}^{L^2} \sum_{uv} T^t(Lx - u) \nabla_u c_{uv}(t) T^{L^2-t-1}(v - Ly) + \gamma_{xy}.$$

where  $\gamma$  involves quadratic and higher order polynomials in  $c$ .

Ignoring first  $r$  we get for the average flow

$$T_n = L^{nd} T^{L^{2n}}(L^n \cdot)$$

i.e.

$$\hat{T}_n(k) = \hat{T}^{L^{2n}}(k/L^n).$$

Write  $\hat{T}(k) = 1 - ck^2 + o(k^2)$ . Then as  $n \rightarrow \infty$ :

$$\hat{T}_n(k) \rightarrow e^{-ck^2}$$

explaining the fixed point.

Similarly, ignoring  $\gamma$  the noise is driven by the linear map

$$\mathcal{L}c_{xy}(0) = L^{d-1} \sum_{t=1}^{L^2} \sum_{uv} T^t(Lx - u) c_{uv}(t) T^{L^2-t-1}(v - Ly).$$

The variance of  $\mathcal{L}c$  contracts:

$$(31) \quad \mathbb{E}(\mathcal{L}c)^2 \sim L^{-d} E c^2.$$

The intuitive reason behind this is the following. Take e.g.  $x = y = 0$ . For  $t$  of order  $L^2$ ,  $T^t(Lx - u) \sim L^{-d} e^{-|x-u/L|}$ . Hence the  $u$  and the  $v$  sums are localized in an  $L$  cube at origin. Since  $c_{uv}(t)$  has exponential decay in  $|u - v|$

$$(32) \quad \mathcal{L}c_{00}(0) \sim L^{-d-1} \sum_{t=1}^{L^2} \sum_{|u|<L} c_{uu}(t).$$

Since correlations of  $c$  decay exponentially in space and time (32) is effectively a sum of  $L^{d+2}$  independent random variables of variance  $L^{-2d-2}(\mathbb{E}c)^2$  thus leading to (31).

Taking into account the corrections  $r$  and  $\gamma$  in (29) and (30) we conclude that the variance contracts as

$$\mathbb{E}(c_n)^2 \sim \epsilon_n = L^{-nd}\epsilon.$$

The iteration of the mean becomes

$$(33) \quad T_{n+1} = L^d T_n^{L^2}(L \cdot) + \mathcal{O}(\epsilon_n).$$

The fixed point is the same but the  $\mathcal{O}(\epsilon_n)$  renormalizes the diffusion constant  $\kappa$  at each iteration step (less and less as  $n \rightarrow \infty$ ).

There is a problem however once we try to make this perturbative analysis rigorous. Deterministically the noise is *relevant*: from (32) we see that  $\|\mathcal{L}c\|_\infty$  can be as big as  $\mathcal{O}(L)\|c\|_\infty$ . This means that there are unlikely events in the environment where the random walk develops a drift. We write

$$|c_n(t, \cdot, E)| \leq L^{N_n(x)-bn}.$$

Then  $N_n(x)$  can be (very) large, but with (very) small probability:

$$\text{Prob}(N_n(x) > N) \leq e^{-KN}$$

with  $K$  large.

Finally, to control the *nonlinear* contributions to  $\Phi_n$  we show that the second derivative  $D_E^2 \Phi$  is irrelevant in all dimensions due to the scaling of  $E$ :

$$\mathcal{R}\Phi(t, x, E) = L^d(\Phi(t_{L^2}) \circ \dots \circ \Phi(t_1))(Lx, L^{-d}E(\cdot/L)).$$

## 9. TOWARDS HAMILTONIAN SYSTEMS

The coupled map lattices we have discussed are an alternative microscopic model with a local conservation law that under a macroscopic limit gives rise to diffusion. To be realistic they should however share some features with the Hamiltonian systems that are more familiar and physically relevant. From this point of view there is a lot missing from our analysis.

The first problem to understand is to go beyond the perturbative analysis around  $E = 0$  (i.e. zero temperature). Then the equation (24) picks also a driving term.

The second unnatural assumption is the  $E$ -independence of the  $\theta$  dynamics. In a realistic model rare configurations of  $E$  can slow down the  $\theta$  dynamics. Also the annealed system is probably not uniformly elliptic as we assumed and the random drift can create traps in the environment with long lifetimes.

All these issues can and should be studied with the renormalization group approach sketched above.

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